Zahra Solatideh kordi

Exam I MTH 221, Fall 2016

Ayman Badawi

$\frac{68}{70}$ Excellent

QUESTION 1. Given $A = \begin{bmatrix} 2 & 2 \\ -3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} \sqrt{2} & 3 & 2 \\ 2 & 1.3 & -1 \end{bmatrix}$. Let $C = AB$.

(i) Use the method of linear combination of columns to find the third column of C.

$$col_3(C) = 2\begin{bmatrix} 2 \\ -3 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$$

(ii) If A is invertible (nonsingular), then find A^{-1}.

$$d = 8 - (-6) = 8 + 6 = 14$$

$$A^{-1} = \frac{1}{14}\begin{bmatrix} 4 & -2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{14} & \frac{-2}{14} \\ \frac{3}{14} & \frac{2}{14} \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & \frac{-1}{7} \\ \frac{3}{14} & \frac{1}{7} \end{bmatrix}$$

QUESTION 2. Find a matrix 2×3 such that $A\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$.

$$A\left(\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - I_3\right) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$B^{-1} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_3+R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$A\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I_3} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

QUESTION 3. Given A is a 4×4 matrix such that $A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$. Find the solution set to the system of linear equations $A^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

$$\underbrace{(A^T)^{-1} A^T}_{I_4} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{(A^T)^{-1}}_{(A^{-1})^T} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 8 \end{bmatrix}$$

solution set : $\{(1, 0, 0, 8)\}$

QUESTION 4. Let $A = \begin{bmatrix} 1 & 1 & -3 \\ -1 & 0 & 3 \\ -1 & 0 & 4 \end{bmatrix}$

(i) Find A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ -1 & 0 & 3 & 0 & 1 & 0 \\ -1 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow[-R_2+R_1 \to R_1]{-R_2+R_3 \to R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \xrightarrow{3R_3+R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -4 & 3 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 0 & -4 & 3 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

(ii) Find $(A^T)^{-1}$

$$(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} 0 & 1 & 0 \\ -4 & 1 & -1 \\ 3 & 0 & 1 \end{bmatrix}$$

(iii) Find $|A|$, $|A^{-1}|$, and $|A + A^{-1}|$

$$|A| \overset{\text{row } 3}{=} (-1)^2(-1)\begin{vmatrix} 1 & -3 \\ 0 & 3 \end{vmatrix} + (-1)^6 4 \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} = -(3) + 4(0-(-1)) = -3+4 = 1$$

$|A| = 1$

$|A^{-1}| = 1$

$$A + A^{-1} = \begin{bmatrix} 1 & 1 & -3 \\ -1 & 0 & 3 \\ -1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -4 & 3 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ -1 & -1 & 5 \end{bmatrix}$$

$$|A+A^{-1}| \to \text{row } 2: \quad (-1)^4 \begin{vmatrix} 1 & 0 \\ -1 & 5 \end{vmatrix} + (-1)^5 3 \begin{vmatrix} 1 & -3 \\ -1 & -1 \end{vmatrix} = 5 - 3(-1-3) = 5 - 3(-4) = \boxed{5+12 = 17}$$

(iv) Find $|3A|$ and $|A^3|$

$|3A| = 3^3 \times 1 = 27$ $\quad |A^3| = |A \times A \times A| = |A||A||A| = 1$

(v) View each row of A as a point in R^3 and let $D = span\{R_1, R_2, R_3\}$. Find $dim(D)$.

-1

$\dim(D) = \cancel{R^3}\ 3$ $\quad$ 3

because A invertible $\to$ $|A| \neq 0 \to$ rows are independent

$D = \mathbb{R}^3$

QUESTION 5. Given $A = \begin{bmatrix} 2 & a & b \\ c & 3 & 7 \\ d & -1 & 3 \end{bmatrix}$ and $|A| = -7$ where a, b, c, d are some numbers.

(i) Let $B = \begin{bmatrix} 10 & a & b \\ c & 3 & 7 \\ d & -1 & 3 \end{bmatrix}$. Find $|B|$. [Hint: Use the first row of A and the first row of B to find $|A|$ and $|B|$, then somehow a miracle will be observed!]

$$|A| = (-1)^2 2 \begin{vmatrix} 3 & 7 \\ -1 & 3 \end{vmatrix} + (-1)^3 a \begin{vmatrix} c & 7 \\ d & 3 \end{vmatrix} + (-1)^4 b \begin{vmatrix} c & 3 \\ d & -1 \end{vmatrix} = -7$$

$$|B| = (-1)^2 10 \begin{vmatrix} 3 & 7 \\ -1 & 3 \end{vmatrix} + (-1)^3 a \begin{vmatrix} c & 7 \\ d & 3 \end{vmatrix} + (-1)^4 b \begin{vmatrix} c & 3 \\ d & -1 \end{vmatrix}$$

$$|A| = 2(16) - a(3c - 7d) + b(-c - 3d) = 32 - 3ac + 7ad - bc - 3db = -7$$

$$|B| = 10(16) - a(3c - 7d) + b(-c - 3d) = 160 - 3ac + 7ad - bc - 3db =$$

$$= 128 + \underbrace{32 - 3ac + 7ad - bc - 3db}_{-7}$$

$$\Rightarrow \boxed{|B| = 128 - 7 = 121}$$

(ii) Let $C = \begin{bmatrix} 4 & 2a & 2b \\ d+2 & a-1 & b+3 \\ c & 3 & 7 \end{bmatrix}$. Find $|C|$.

$$\overset{A}{\begin{bmatrix} 2 & a & b \\ c & 3 & 7 \\ d & -1 & 3 \end{bmatrix}} \xrightarrow{2R_1} \overset{E}{\begin{bmatrix} 4 & 2a & 2b \\ c & 3 & 7 \\ d & -1 & 3 \end{bmatrix}} \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\underset{F}{\begin{bmatrix} 4 & 2a & 2b \\ d & -1 & 3 \\ c & 3 & 7 \end{bmatrix}} \xrightarrow{\frac{1}{2}R_1 + R_2 \to R_2} \underset{C}{\begin{bmatrix} 4 & 2a & 2b \\ d+2 & a-1 & b+3 \\ c & 3 & 7 \end{bmatrix}}$$

$|E| = 2|A| \rightarrow |E| = -14$

$|F| = -|E| \rightarrow |F| = 14$

$|C| = |F| \rightarrow \boxed{|C| = 14}$

QUESTION 6. Find the solution set to the system :

$$x_1 + x_2 + x_3 + x_4 = 4$$

$$-x_1 - x_3 - x_4 = 10$$

$$-2x_1 - 2x_2 - 2x_3 - 2x_4 = -8$$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & c \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ -1 & 0 & -1 & -1 & 10 \\ -2 & -2 & -2 & -2 & -8 \end{array}\right] \xrightarrow[2R_1 + R_3 \to R_3]{R_1 + R_2 \to R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 0 & 14 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-R_2 + R_1 \to R_1}$$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & c \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & -10 \\ 0 & 1 & 0 & 0 & 14 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_3 + x_4 = -10$

$x_2 = 14$

$x_1 = -10 - x_3 - x_4$

Solution set : $\{(-10 - x_3 - x_4, 14, x_3, x_4) \mid x_3, x_4 \in \mathbb{R}\}$

QUESTION 7. Given the augmented matrix of a system of linear equations $A = \left[\begin{array}{ccc|c} 1 & -a & 3 & 4 \\ -1 & 1+a & -3 & -2 \\ -1 & a & b & c \end{array}\right]$

(i) For what values of a, b, c will the system have unique solution?

$$\left[\begin{array}{ccc|c} 1 & -a & 3 & 4 \\ -1 & 1+a & -3 & -2 \\ -1 & a & b & c \end{array}\right] \xrightarrow[R_1+R_3\to R_3]{R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & -a & 3 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & b+3 & c+4 \end{array}\right]$$

unique solution:

$$b \neq -3$$

$$\left.\begin{array}{l} a \in \mathbb{R} \\ c \in \mathbb{R} \end{array}\right\} \text{can be any number}$$

(ii) For what values of a, b, c will the system have infinitely many solutions?

$b = -3$

$c = -4$

$a =$ any number $\in \mathbb{R}$

QUESTION 8. Given A is a 4×4 matrix such that $A \xrightarrow{R_1 \leftrightarrow R_3} B \xrightarrow{2R_4} C = \begin{bmatrix} 1 & -4 & -2 & 8 \\ -1 & 6 & 2 & -6 \\ -1 & 4 & 2 & -6 \\ -1 & 4 & 3 & -8 \end{bmatrix}$

1) Find $|A|$.

$$\overset{C}{\begin{bmatrix} 1 & -4 & -2 & 8 \\ -1 & 6 & 2 & -6 \\ -1 & 4 & 2 & -6 \\ -1 & 4 & 3 & -8 \end{bmatrix}} \xrightarrow[R_1+R_4 \to R_4]{\substack{R_1+R_2 \to R_2 \\ R_1+R_3 \to R_3}} \overset{D}{\begin{bmatrix} 1 & -4 & -2 & 8 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}} \xrightarrow{\frac{1}{2}R_2}$$

$$\overset{E}{\begin{bmatrix} 1 & -4 & -2 & 8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}} \xrightarrow{R_3 \longleftrightarrow R_4} \overset{F}{\begin{bmatrix} 1 & -4 & -2 & 8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}}$$

$|B| = -|A|$

$|C| = 2|B| = -2|A|$

$|D| = |C| = -2|A|$

$|E| = \frac{1}{2}|D| = -|A|$

$|F| = |E| = -|A|$

$|G| = -|F| = |A|$

$|G| = 2 \to |A| = 2$

2) Find A

$$\begin{bmatrix} 1 & -4 & -2 & 8 \\ -1 & 6 & 2 & -6 \\ -1 & 4 & 2 & -6 \\ -1 & 4 & 3 & -8 \end{bmatrix} \xrightarrow{\frac{1}{2}R_4} \begin{bmatrix} 1 & -4 & -2 & 8 \\ -1 & 6 & 2 & -6 \\ -1 & 4 & 2 & -6 \\ -\frac{1}{2} & 2 & \frac{3}{2} & -4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} -1 & 4 & 2 & -6 \\ -1 & 6 & 2 & -6 \\ 1 & -4 & -2 & 8 \\ -\frac{1}{2} & 2 & \frac{3}{2} & -4 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 4 & 2 & -6 \\ -1 & 6 & 2 & -6 \\ 1 & -4 & -2 & 8 \\ -\frac{1}{2} & 2 & \frac{3}{2} & -4 \end{bmatrix}$$

QUESTION 9. Given D = span{(1, 1, 1, 1), (−1, 0, 0, 0), (0, 1, 1, 1), (−1, 1, 1, 1)}.

1. Find $dim(D)$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_1+R_4\to R_4]{R_1+R_2\to R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \end{bmatrix} \xrightarrow[-2R_2+R_4\to R_4]{-R_2+R_3\to R_3} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\dim(D) = 2$$

2. In view of your answer to part (1), does $D = R^4$? why? No because it only has two independent points, in order to be R^4 it needs to have 4 independent points.

3. Convince me that the point (2, 8, 8, 8) live inside D.

$$0(1,1,1,1) - 2(-1,0,0,0) + 8(0,1,1,1) + 0(-1,1,1,1) =$$

$$(2,0,0,0) + (0,8,8,8) = (2,8,8,8)$$

→ (2, 8, 8, 8) is a linear combination of the points therefore it lives in D.

another way: Now we know $D = span\{(1,1,1,1), (-1,0,0,0)\}$

so check: $(2,8,8,8) = \alpha_1(1,1,1,1) + \alpha_2(-1,0,0,0) \to$ Find $\alpha_1, \alpha_2 \to \alpha_1 = 8, \alpha_2 = -2$

4. Does the point (2, 5, 6, 6) live in D?Explain? No because it cannot be written as a linear combination of the points (1,1,1,1), (−1,0,0,0), (0,1,1,1), (−1,1,1,1)

show why not

same as above

$(2,5,6,6) = \cancel{\alpha_1(1,1,1,1) + \alpha_2(-1,0,0,0)}$
$= \alpha_1(1,1,1,1) + \alpha_2(-1,0,0,0)$

$2 = \alpha_1 - \alpha_2$
$5 = \alpha_1 \to \alpha_1 = 5, \alpha_2 = 3$

Faculty information

Ayman Badawi, Department of Mathematics & Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates.
E-mail: `abadawi@aus.edu`, `www.ayman-badawi.com`

$(2,5,6,6) \neq 5(1,1,1,1) + 3(-1,0,0,0)$

$\notin D$